Hysteresis and metastability in the quenched turbulent dynamics of the complex Ginzburg-Landau equation

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(Received 5 July 2001; published 19 December 2001)

We consider the quenched dynamics of the two-dimensional complex Ginzburg-Landau equation in its turbulent regime. We initialize the system in a frustrated state and observe how frustration affects the evolution towards the turbulent state. This process is performed for parameter values where, for random initial conditions, the system evolves into the turbulent state. We observe that the glassiness of the initial condition can inhibit the occurrence of the absolute instability close to the critical point for that instability in parameter space. Sufficiently far from the critical point, the turbulent state will develop, but only after spending considerable time in a transient metastable state of fixed vortex density. The parameter distance from the critical point is found to scale as an exponential of a power of the lifetime of the metastable state, and with a power exponent depending on the "depth" of the original quench. The limiting regimes of shallow and deep quench are identified by their respective values of the exponent, and the distinct mechanisms leading to the relaxation to turbulence in each case are highlighted.

DOI: 10.1103/PhysRevE.65.016122 PACS number(s): 64.60.Cn, 64.60.My, 05.45.−a, 64.60.Qb

A ubiquitous equation in the study of pattern formation in oscillatory, nonequilibrium media is the complex Ginzburg-Landau equation (CGL) [1–4]. This equation has been recognized as essential to the study of slow modulations of oscillations in a continuous medium near a Hopf bifurcation threshold [5]. Furthermore, it is a robust model for the description of the dynamics of point defects in Rayleigh-Bènard convection [6], as well as a simple description for the behavior of spiral waves in systems such as the Belousov-Zhabotinsky (BZ) reaction [7].

Extensive studies have been devoted to the dynamical scenarios in the different regimes of parameter space of the two-dimensional (2D) CGL (see for example Refs. [5,8] and references therein). We will consider here the CGL in the form

$$\frac{\partial A}{\partial t} = A + (1+ib)\Delta A - (1+ia)|A|^2 A, \qquad (1)$$

where A is a complex field and all other parameters are real. The possible instabilities (Benjamin-Feir, Eckhaus, convective, and absolute 2D instabilities) are well documented (see e.g., Ref. [8]). It is also well known that the 2D CGL exhibits three basic types of behavior (strongly suggested by numerical experiments and in reasonable agreement with existing theory), depending on parameter values:

(1) The system can be in a defect "turbulent" state in which topological defects (spiral waves or vortices) are continuously generated and destroyed in pairs, thus preserving the total topological charge.

(2) For a large range of parameter values, no vortices can be sustained in the asymptotic long time state, so eventually all vortex pairs annihilate, leaving a quiescent pattern [8]. This regime is often referred to as being phase turbulent [5].

(3) There is an intermediate "vortex glass" regime in which metastable cellular patterns emerge. In this long lived non-equilibrium state, the vortices in a simulation [see, e.g., panel (a) of Fig. 1] can be seen to tremble slightly, however, they persist for very long times. One can also clearly discern so-called "shock lines" defining the boundaries of the cells. These shocks are local maxima of the norm field ($\rho=|A|$) created by the interference of the plane waves emitted by the defect cores [9–11].

It is known that the transition to the defect turbulent state occurs through an absolute instability (AI), see, e.g., Ref. [8]. For random initial conditions (IC), this means that the instability is abruptly manifested beyond a critical point $a_{c,1}$, and for $a \geqslant a_{c,1}$ the system randomly generates and annihilates defect pairs. For $a < a_{c,1}$, random IC eventually relax into a frozen so-called "vortex-glass" state. It should be noted that the "glassiness" in this case, rather than being created by disorder as in, e.g., spin-glass models, is a result of the intrinsic nonlinearity and competing length scales of the model. Hence, as is also pointed out in a very interesting recent review paper on the subject [12], it is of interest to seek an understanding of the effects of such nonlinearity-induced frustration in the dynamics of the system. The second motivating factor for this work is the study of the inverse problem in Ref. [13]. The authors of Ref. [13] considered the situation where a turbulent state (created numerically within the turbulent regime) is used as IC within the glassy regime. They were able to show that below a critical point in the glassy state regime of parameters, the system relaxes directly to the vortex glass. Beyond the critical point, it supports a metastable quasiturbulent state that eventually relaxes to a cellular pattern. The logarithm of the lifetime of this metastable state was found to scale as a power of the distance in parameter space from the critical point.

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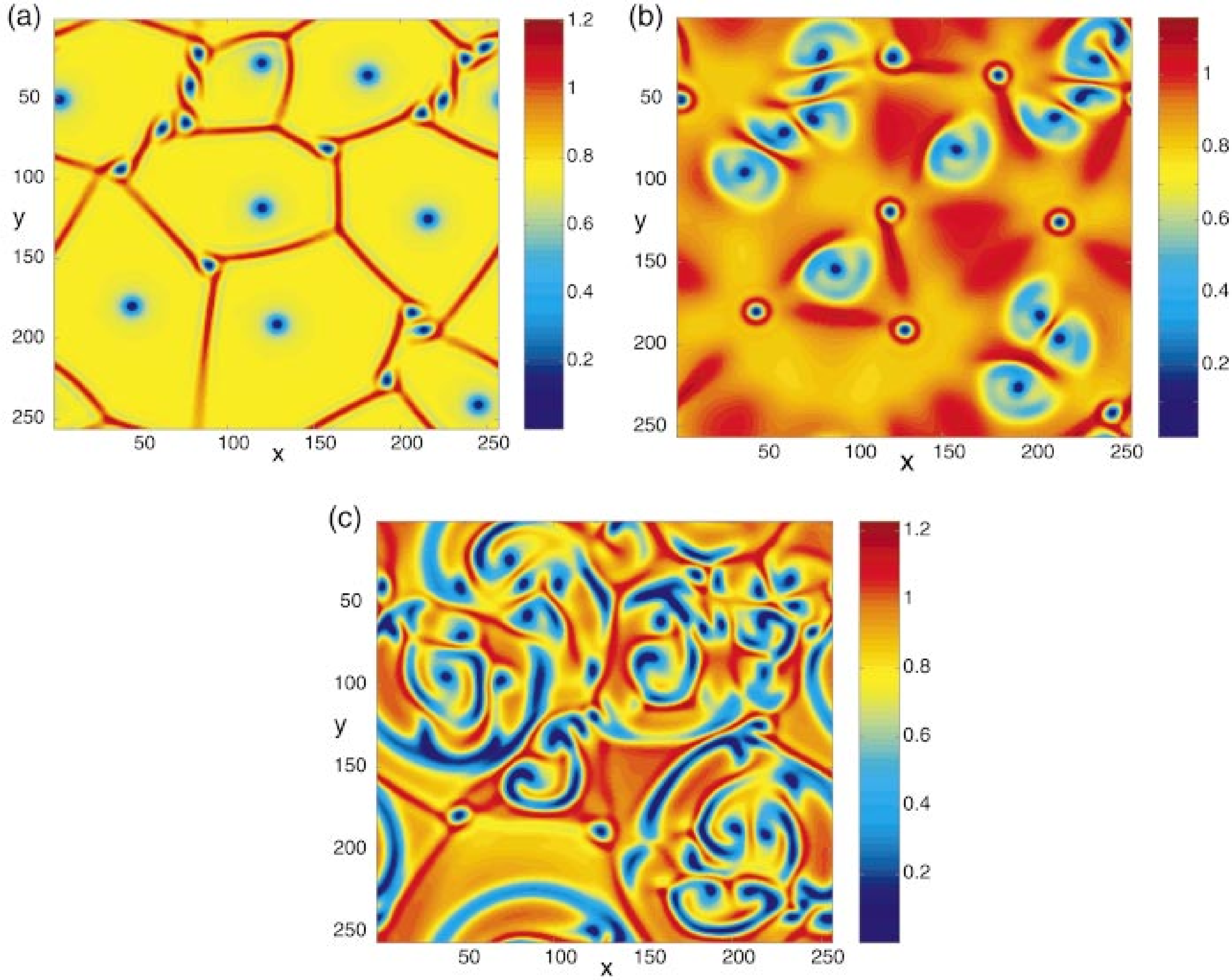

FIG. 1. (Color) Panel (a): frustrated 2D vortex pattern for $a=0.65$, $b=-0.75$. Shown is the density plot of the norm field $\rho=|A|$ of the solution of Eq. (1) after the system has relaxed from its random initial conditions to the ground state "vortex glass" configuration. Panel (b): the previous configuration is now shallowly quenched to $a=0.9$ and the frustrated initial condition will eventually "melt." This panel shows a snapshot during this relaxation process. Panel (c): here the resulting turbulent state is shown a long time after the initial quench. These figures correspond to the regime of $\delta\approx 1$ (shallow quench).

Here we investigate the effects of initial frustration on the turbulent dynamics. Specifically, we conduct numerical simulations within the intermediate regime of parameter values and obtain frozen vortex configurations. Then, we initialize the system with these configurations as IC but within the turbulent regime of parameter space and follow the ensuing dynamics. For our numerical studies of the time evolution, we have used fourth order explicit and implicit methods. The boundary conditions are periodic in both the spatial variables x and y. The vortices are identified as local minima of the norm field in the neighborhood of its zeros [14] and the numerical calculations are performed along a horizontal line in the (a,b) parameter space. It should also be noted that all of the presented results are for $a>0$. These can be suitably interpreted for values of $a<0$ using the symmetries of the model. Hence, we set $b=-0.75$ and vary a. For random IC we find that the AI occurs at $a_{c,1}\approx 0.855$ [15].

We now initialize the system with random IC for $a_{in}=0.75,0.65,0.55,\ldots,0.15$. The resulting final configuration is a frozen cellular pattern of the type shown in the top left panel of Fig. 1, for, i.e., $a_{in}=0.65$. We will use the subscript in for the initial (i.e., the equilibrium frustrated configuration for this parameter value that will subsequently be used as IC in the quenching simulations). For the quenched values of a we will use the subscript *quen*. Quenching the frozen states of $b=-0.75$, $a=a_{in}$ to parameter values $b=-0.75$, $a_{quen}>a_{c,1}$, we observe the following:

(1) For $a_{quen}<a_{c,2}$ there is a hysteretic inhibition of the turbulent phase. In particular, for $a_{quen}\in(a_{c,1},a_{c,2})$ in our simulations of $O(10^4–10^5)$ time units, the system does not develop the defect turbulent state. $a_{c,2}$ is dependent on the "depth" of the initial quench, i.e., how far the original prequenched value a_{in} is from $a_{c,1}$. As expected and documented by the data of Table I, if the system is frustrated for "nearby" parameter values (shallow quench), then it will be more difficult for it to avoid the frustration and identify its

TABLE I. Critical values $a_{c,2}$ for different a_{in}.

a_{in}	0.75	0.65	0.55	0.45	0.35
$a_{c,2}$	0.875	0.875	0.865	0.855	0.855

ground state [16]. Hence for a shallow quench, the hysteretic interval is larger. As the quench deepens, the situation gradually approaches the one with random IC and the system is able to identify the AI already at $a_{quen} \approx a_{c,1}$, without hysteretic effects. We conclude that initial frustration can freeze the system into the glassy state, even when its ground state is turbulent, displacing in parameter space the point of manifestation of the AI.

(2) For $a_{quen} \geqslant a_{c,2}$, the system will eventually be able to avoid the frustration induced by the IC and reach the turbulent ground state configuration. However, the initial glassiness still has an effect on the dynamics. As is well known, in the turbulent state the number of vortices in the system is very high and continuously fluctuating, while the same is not true for the fixed vortex density glassy configurations. Probing the time evolution of the vortex number in the system for the case at hand (a typical result is shown in Fig. 2), We observe that the initial frustration induces a fixed number of vortices for a period of time. The time interval T for which the vortex number remains at the original value determined by the IC, can be interpreted as the duration of a metastable state. Such a metastable state is not present for random IC, in which case a large number of vortices arise from the IC almost immediately. This metastable state can be explained as the by-product of the initial frustration of the configuration that necessitates an interval of time prior to the relaxation of the system to its ground state, and its overcoming the glassiness in favor of this turbulent ground state. Once again the deeper the quench, the easier it becomes for the system to find its way out. This can be interpreted in two ways. For a fixed IC-related a_{in}, say 0.65, for $a_{quen,1} > a_{quen,2}$ with $a_{quen,1;2} > a_{c,2}$, $T_2 > T_1$. But also for a fixed $a_{quen} > a_{c,2}$ for two different $a_{in,1} > a_{in,2}$ IC-related values of a_{in} ($a_{in,1;2} < a_{c,2}$), $T_{a_{quen},a_{in,1}} > T_{a_{quen},a_{in,2}}$. The above notions suggest the interpretation of the parameter a as a ‘‘temperature’’ for the system. In this perspective, for a sufficiently energetic ‘‘thermal’’ quench, the system has enough ‘‘energy’’ to overcome the disorder-induced ‘‘energy barriers’’ and reach its ground state. It should also be emphasized that deeper quenches imply larger jumps in the steady value of the modulus of the selected spiral waves. This simple argument can also be used to justify the observed differences between shallow and deep quenches.

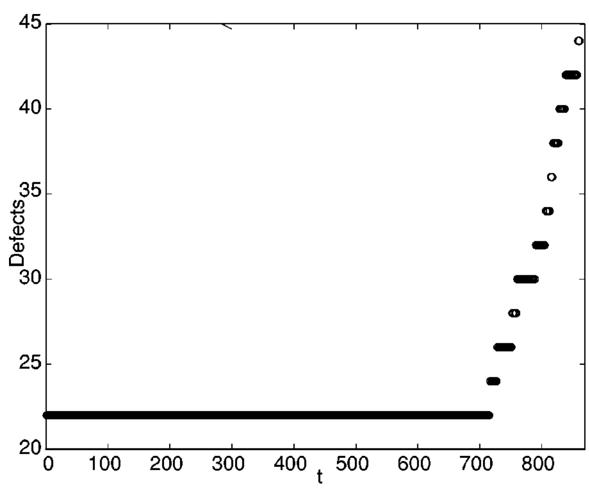

FIG. 2. The number of vortices as a function of time for a typical quenching process beyond $a_{c,2}$. For a well-determined period of time T (this is how T is defined), the system is in the frustrated metastable state. Eventually, however, it relaxes to the ground state of defect turbulence.

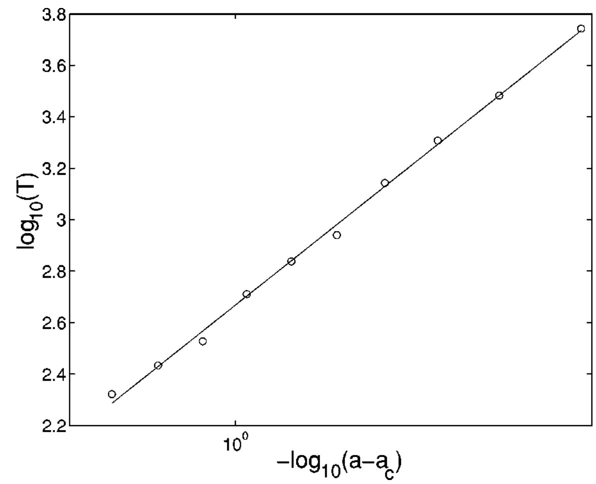

FIG. 3. Plot of $\log_{10}(T)$ as a function of $-\log_{10}(a-a_c)$ in a semilogarithmic plot. The slope s of this line is $s=1/\delta$ and the intercept $p=\log_{10}(\tau)$.

(3) The distance of a_{quen} from the (new) critical point $a_{c,2}$ scales as an exponential of a power of the lifetime of the metastable state. This type of relaxation behavior has been observed in a number of glassy systems such as frustrated Josephson junction arrays (see, e.g., Ref. [17]) or semiconducting materials (see, e.g., Refs. [18,19]). In particular,

$$a_{quen} - a_{c,2} \sim \exp[-(T/\tau)^{\delta}]. \tag{2}$$

A typical manifestation of this scaling is shown in Fig. 3 for quenching from $a_{in}=0.65$. The values of δ, τ depend again on the ‘‘depth’’ of the initial quench. For the different values of $a=a_{in}$, the corresponding fitted values of δ are given in Table II. It can be seen that δ increases as a_{in} is lowered (for

TABLE II. Values of δ for different a_{in}.

a_{in}	δ
0.75	0.998
0.65	0.998
0.55	1.123
0.45	1.447
0.35	1.986
0.25	2.026
0.15	2.014

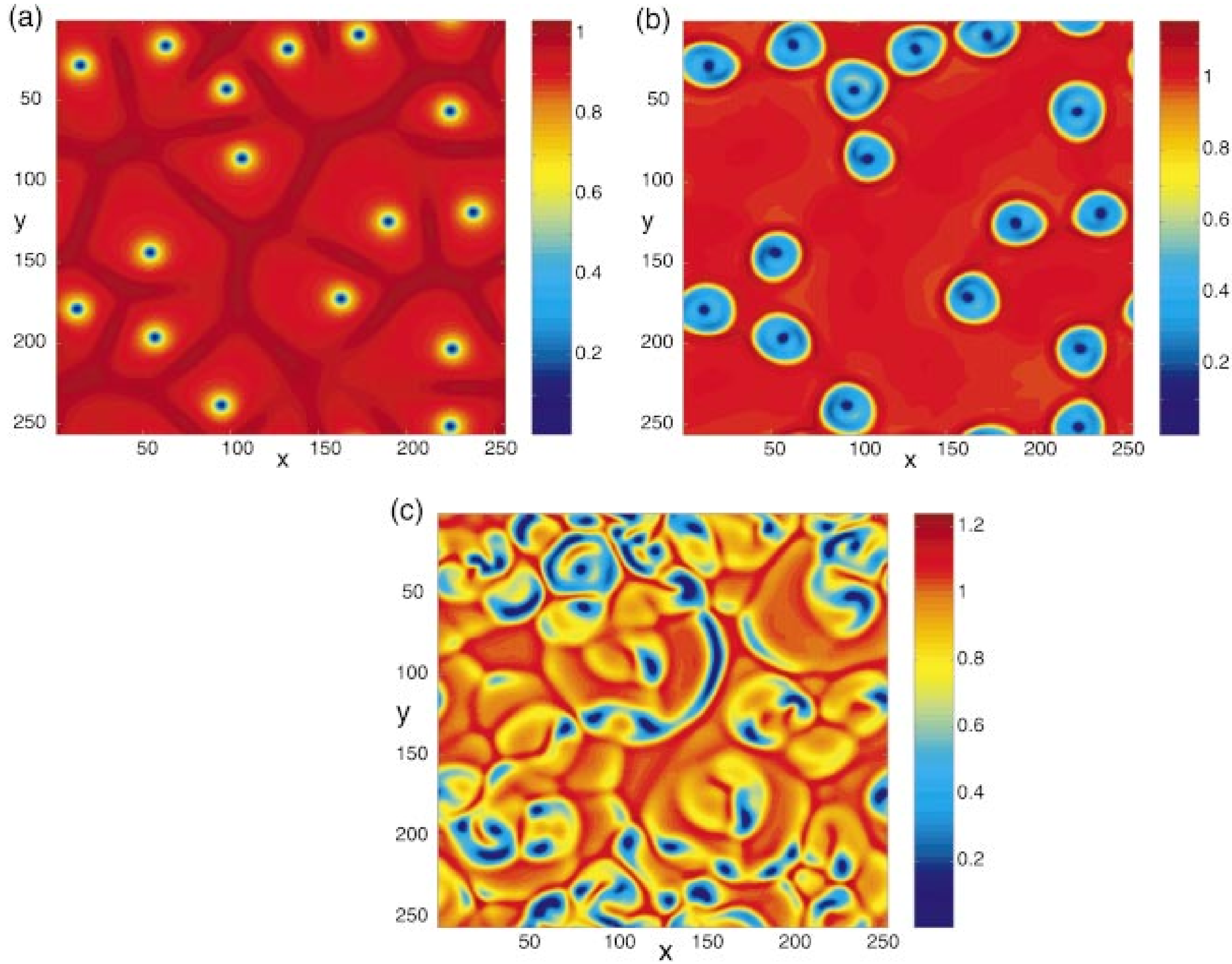

FIG. 4. (Color) The three panels are exactly analogous to those of Fig. 1 but for a deep quench. $a_{in}=0.15$ and $\delta\approx 2$ for this case. It can clearly be seen that the mechanism melting the pattern is different from the shallow quench case of Fig. 1.

decreasing initial frustration). It is interesting to note that δ is always observed to lie between 1 and 2. In fact for the limiting cases of shallow and deep quench, saturation of the value of δ is observed to the corresponding limit ($\delta=1,\delta=2$, respectively). This suggests limits of unimolecular and bimolecular type relaxation mechanisms. In fact, the different limits correspond to different relaxation mechanisms that are the pathways for "melting" the cellular structure in the shallow and the deep quench cases. In the numerical experiments it was also observed that the "half-life" time τ is larger for higher a_{in} (data not shown). The monotonic behavior here may not persist in the saturation limit. However, as the parameters are varied between the saturation limits (from shallow to deep quench), the variation is indeed monotonic. The above conclusions are in qualitative agreement with the intuition that the shallower the quench, the more effective frustration will be in preventing the system from reaching its ground state, in agreement with the comments in the previous paragraph. It should be noted here that the behavior of Eq. (2) is a direct result of the initial frustration. An initial perturbation (for random or nonfrustrated IC) will grow beyond the AI critical point as χ^t. This signifies that the time needed for entering the turbulent regime is $\delta t\sim|\log_{10}\chi|^{-1}$. A leading order expansion shows [13] that $\log_{10}\chi\sim(a-a_c)$. Hence one expects, based on this analysis, that

$$a-a_c\sim T^{-1}, \tag{3}$$

for a random initial condition. The exponential behavior and the delay, both in the manifestation of the AI as well as in the identification of the turbulent ground state, can thus be naturally attributed to the glassiness inherent to the IC of the system.

The elementary mechanism by which turbulence eventually "melts" the initial cellular structure is shown in Figs. 1 and 4. In the case of shallow quench (well-formed cellular structures) defects start forming initially at the corners of the configuration (in the vicinity of the so-called edge defects [11]). Then, in a way reminiscent of the inverse procedure (illustrated by Fig. 3 of Ref. [13]) of the procedure mentioned in Ref. [11], the turbulent rings of vorticity gradually reduce the areas "shielded" by the larger defects. They eventually allow defect turbulence as they suppress the radii of the droplets of larger vortex shielded regions to zero. On the contrary, for a deep quench the initial configuration consists of defect pairs and quartets rather than of a cellular structure with individual defects well separated by shock lines. In this case, the ''melting'' process resembles the process in Ref. [13]. The initially very narrow vortices start growing uniformly. The uniformity can be viewed in two ways. On the one hand all vortices grow. However, the growth appears to be in turbulent droplets of uniformly increasing radius. These eventually overwhelm the pattern, creating the defect turbulent state. The relaxation path between the two saturation regimes follows the variation of the relative influence of these two distinct mechanisms.

We note that even though our results have been obtained along a horizontal line in the (a,b) parameter space, we have found the described scenario to be general for the quenching into the turbulent regime of the 2D CGL. In particular, we performed simulations along the vertical line $a=1.0$, where, for random IC, AI sets in at $b_{c,1}\approx -0.495$. For $b_{in}=-0.4$, inhibition of the transition was observed until $b_{c,2}\approx -0.515$. Also, scaling similar to the one shown in Fig. 3 was observed. The existence of such a general scenario is consistent with the implications of the existence of a similarity transformation [given by Eq. (47) in Ref. [2] for the relevant linearized homogeneous equation] which transforms the results for the case of $b\neq 0$, to the case $b=0$ with a new $\tilde{a}=(a-b)/(1+ab)$. This is merely an argument in support (but, by all means, not a proof) of the genericity of the described scenario. We conclude that frustrated initial conditions can affect the relaxation to the turbulent state. In particular, they can move the critical point, causing a hysteretic effect in the occurrence of the transition. Even when the transition does eventually take place, these effects temporally delay its appearance compared to the random or nonfrustrated IC case by inducing a metastable state of fixed vorticity. Frustration is manifested through an exponential behavior of the parameter distance from the critical point as a function of a power of the lifetime of the metastable state. The exponent of the power depends on the ''depth'' of the initial quench, saturating to limiting values for shallow and deep quenches. It would be of interest to identify and follow more closely further characteristics of this nonlinear glassy system, to understand how nonlinearity and length scale competition can induce ''disorder.'' In particular, studies of the possible effective thermodynamics of the vortex motion, or of variations in the frozen state as a function of the system parameters, are interesting topics for future work.

P.G.K. gratefully acknowledges support from the Alexander S. Onassis Public Benefit Foundation. Research at the Los Alamos National Laboratory has been performed under the auspices of the US DOE under Contract No. W-7405-ENG-36.

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[14] Observation of the norm field indicates that in general $|A|$ is in the vicinity of the steady state, apart from the shock lines (where it obtains larger values) and abrupt dips of the field close to zero, which constitute the vortices. We observe the values of the field close to zero (i.e., below a judiciously selected value of $|A|$) and by interpolation and/or averaging methods we reconstruct the local minima of the field. This methodology along with the monotonicity of the field in the neighborhood of the dips ensure precise determination of the number of vortices.
[15] Let us indicate here what we will mean by this notation: for $a=0.85$ the system will freeze into a glassy configuration, while for $a=0.86$ it will reach a turbulent state. This signifies that the transition takes place in the interval $(0.85, 0.86]$ and one can, in principle, identify the exact critical point. However, since detailed simulations and possibly very long computing times may be needed to determine the exact critical point without adding to the physical intuition, in the following we will *set* the critical point to 0.855 with the understanding of the approximation involved.
[16] By ground state we refer to the state to which the system will relax in the long time asymptotics (starting from random IC, unless otherwise indicated).
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